- What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

Given, $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

We know that the principal branch for $\cos^{-1}$ is $[0, \pi]$ and for $\sin^{-1}$ it is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3} + \sin^{-1}\left(\sin\left(\pi - \frac{2\pi}{3}\right)\right) = \frac{2\pi}{3} + \pi - \frac{2\pi}{3} = \pi$$

This is between $[0, \pi]$

Since, $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we subtract it from π to bring $\sin^{-1}$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- Evaluate $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

$$= \cos 15^\circ \cos 75^\circ - \sin 75^\circ \sin 15^\circ$$

[notice it is of the form $\cos A \cos B - \sin A \sin B$] $= \cos(A+B)$

$$= \cos(15^\circ + 75^\circ) = \cos 90^\circ = 0$$

- Prove that $\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$,

where $-\frac{1}{\sqrt{2}} \le x \le 1$.

Well, this is easy. Let's start with the LHS.

LHS

$$\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right] = \tan^{-1}\left[\frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x})}{(\sqrt{1+x} + \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x})}\right]$$

Rationalize

$$= \tan^{-1}\left[\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}\right] = \tan^{-1}\left[\frac{\left(\sqrt{1+x}\right)^2 - 2\cdot\sqrt{(1+x)(1-x)} + \left(\sqrt{1-x}\right)^2}{1+x+x-1}\right]$$

$$= \tan^{-1}\left[\frac{1+\cancel{x}+1-\cancel{x}-2\sqrt{1-x^2}}{2x}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{2}\left(1-\sqrt{1-x^2}\right)}{\cancel{2}x}\right] = \tan^{-1}\left[\frac{1-\sqrt{1-x^2}}{x}\right]$$

Let, $x = \cos\theta$, $\theta = \cos^{-1}x$, $0 \le \theta \le \frac{3\pi}{4}$, Then

$$\tan^{-1}\left[\frac{1-\sqrt{1-\cos^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\frac{1-\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}{\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}\right]$$

[Half angle formula for sin, cos]

$$= \tan^{-1}\left[\frac{\left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$$

[$\tan\frac{\pi}{4} = 1$]

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}\right]$$

[$\tan(A-B)$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{\theta}{2}, \quad 0 \le \theta \le \frac{3\pi}{4}$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x = \text{RHS.}$$

Hence proved.

- Prove that $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function on $\left[0, \frac{\pi}{2}\right]$

This basically means, we need to show $\dfrac{dy}{d\theta} > 0$, in $\theta \in \left[0, \frac{\pi}{2}\right]$.

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\frac{4\sin\theta}{(2+\cos\theta)}\right] - \frac{d\theta}{d\theta}$$

$$= \frac{(2+\cos\theta)\frac{d}{d\theta}(4\sin\theta) - 4\sin\theta\frac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{(2+\cos\theta).4.\cos\theta - 4\sin\theta.(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2} = \frac{8\cos\theta + 4 - (4 + \cos^2\theta + 4\cos\theta)}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

This is +ve

$$= \frac{\cos\theta\,(4-\cos\theta)}{(2+\cos\theta)^2} > 0 \quad \text{in} \quad \theta \in [0, \tfrac{\pi}{2}]$$

This is +ve, $\cos\theta$ is < 4, in $\theta \in [0, \frac{\pi}{2}]$

This is +ve, because squared!

Thus, our function is increasing.

- Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

Range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Range of $\sec^{-1}$ is $[0, \pi] - \{\frac{\pi}{2}\}$

$\therefore \tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \left[\pi - \sec^{-1}(2)\right]$

$= \frac{\pi}{3} - \left[\pi - \frac{\pi}{3}\right] = \frac{\pi}{3} - \frac{2\pi}{3}$ $\quad \left[\because \sec^{-1}(2) = \frac{\pi}{3}\right]$

$= -\frac{\pi}{3}$. $\quad \left[\sec^{-1}(-x) = \pi - \sec^{-1}(x)\right]$

- Prove that $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

LHS:

$\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$= \cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{2}{\sqrt{13}}\right)$

$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$= \cos\left(\sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{4}{13}} + \frac{2}{\sqrt{13}}\sqrt{1-\frac{9}{25}}\right)\right)$

$= \cos\left[\sin^{-1}\left(\frac{9}{5\sqrt{13}} + \frac{8}{5\sqrt{13}}\right)\right] = \cos\left[\sin^{-1}\frac{17}{5\sqrt{13}}\right]$

$= \cos\left[\cos^{-1}\left(6/5\sqrt{13}\right)\right] = \frac{6}{5\sqrt{13}} = \text{RHS}$

- Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

Principal branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

" " " $\cot^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \left[\pi - \cot^{-1}(\sqrt{3})\right]$

$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

$= \frac{\pi}{3} - \left[\pi - \frac{\pi}{6}\right] = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

- Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= \tan^{-1}\left[2\sin\left(2\times\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right)\right]$

$= \tan^{-1}\left[2\sin\left(2\times\frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[2.\sin\frac{\pi}{3}\right] = \tan^{-1}\left[2.\frac{\sqrt{3}}{2}\right]$

$= \tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$

- Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

LHS

$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

Let, $\sin^{-1}\left(\frac{3}{4}\right) = \theta$, $\sin\theta = \frac{3}{4}$

$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}$

$= \frac{3/4}{1+\sqrt{1-\frac{9}{16}}} = \frac{3/4}{1+\sqrt{\frac{7}{16}}} = \frac{3/4}{1+\frac{\sqrt{7}}{4}}$

$= \frac{3/4 \cdot 4}{(4+\sqrt{7})} = \frac{3}{4+\sqrt{7}} = \frac{3.(4-\sqrt{7})}{16-7} = \frac{3(4-\sqrt{7})}{9} = \frac{4-\sqrt{7}}{3}$ = RHS.

- solve : $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\cos(\tan^{-1}x) = \sin\left(\sin^{-1}\frac{4}{5}\right)$

or, $\cos(\tan^{-1}x) = \frac{4}{5}$

or, $\tan^{-1}(x) = \cos^{-1}\left(\frac{4}{5}\right)$

or, $\tan^{-1}(x) = \tan^{-1}\left(\frac{3}{4}\right)$

or, $x = \frac{3}{4}$

The solution assumes that we are operating in the principal branch for all functions.

- If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$.

$\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$

or, $\tan^{-1}\frac{x+y}{1-xy} = \frac{\pi}{4}$

or, $\frac{x+y}{1-xy} = \tan\frac{\pi}{4}$

or $\frac{x+y}{1-xy} = 1$, or, $x+y = 1-xy$

or, $\underline{x+y+xy = 1}$.

- If $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

$\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$

or, $\tan^{-1}\left[\frac{\frac{x-2}{x-4} + \frac{x+2}{x+4}}{1 - \frac{(x-2)(x+2)}{(x-4)(x+4)}}\right] = \frac{\pi}{4}$

or, $\tan^{-1}\left[\dfrac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)}\right] = \dfrac{\pi}{4}$

or, $\dfrac{x^2+2x-8+x^2-2x-8}{x^2-16-(x^2-4)} = \tan\dfrac{\pi}{4}$

or, $\dfrac{2x^2-16}{x^2-16-x^2+4} = \tan\dfrac{\pi}{4}$

or, $\dfrac{2x^2-16}{-12} = 1$, or, $2x^2-16=-12$

or, $2x^2 = 4$

or, $x^2 = 2.$, or, $x = \pm\sqrt{2}$

- Solve for x: $\tan^{-1}(x+1)+\tan^{-1}(x-1) = \tan^{-1}\left(\dfrac{8}{31}\right)$

$\tan^{-1}\left[\dfrac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right] = \tan^{-1}\left(\dfrac{8}{31}\right)$

or, $\tan^{-1}\left[\dfrac{2x}{1-(x^2-1)}\right] = \tan^{-1}\left(\dfrac{8}{31}\right)$

or, $\dfrac{2x}{1-x^2+1} = \dfrac{8}{31}$, or $62x = 16-8x^2$

or, $8x^2+62x-16 = 0$

or, $4x^2+31x-8=0$

or, $4x^2+32x-x-8=0$

or, $4x(x+8)-(x+8)=0$

or, $(4x-1)(x+8)=0$

$\therefore \boxed{x=\dfrac{1}{4}}$

or, $x=-8$ / $x=\dfrac{1}{4}$

$x=-8$ is not a valid value because $\tan^{-1}(-7)$ and $\tan^{-1}(-9)$ are not defined in the principal branch.

- Prove the following:

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)=0, \quad (0<xy, yx, zx<1)$$

LHS

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$[\because \tan^{-1}x=\cot^{-1}(\tfrac{1}{x})]$$

$$=\tan^{-1}\left(\frac{x-y}{xy+1}\right)+\tan^{-1}\left(\frac{y-z}{yz+1}\right)+\tan^{-1}\left(\frac{z-x}{zx+1}\right)$$

$$=\tan^{-1}x-\tan^{-1}y+\tan^{-1}y-\tan^{-1}z+\tan^{-1}z-\tan^{-1}x$$

$$=0=\text{RHS}.$$

Hence proved.

- Solve the equation for x: $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$.

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\therefore \quad \sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$$

or, $\sin^{-1}\left[x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}\right]=\cos^{-1}x$

or, $\sin^{-1}\left[x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}\right]=\sin^{-1}\left(\sqrt{1-x^2}\right)$

or, $x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$

or, $x\sqrt{1-1-x^2+2x}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$

or, $x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$

or, $x\sqrt{2x-x^2}-x\sqrt{1-x^2}=0$

or, $x\left(\sqrt{2x-x^2}-\sqrt{1-x^2}\right)=0$

Let, $\cos^{-1}x=\theta$

$\cos\theta=x$

or, $\sqrt{1-\sin^2\theta}=x$

or, $1-\sin^2\theta=x^2$

or, $\sin^2\theta=1-x^2$

or, $\sin\theta=\sqrt{1-x^2}$

or, $x=0$ / $\sqrt{2x-x^2}-\sqrt{1-x^2}=0$

or, $2x-x^2=1-x^2$

or, $x=\frac{1}{2}$

- If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, prove that

$$\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$$

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$$

$$\cos^{-1}\left(\frac{x}{a}\cdot\frac{y}{b}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)=\alpha$$

$$\frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha-\frac{xy}{ab}$$ [Square both sides]

$$\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)=\left[\cos\alpha-\frac{xy}{ab}\right]^2$$

$$1-\frac{y^2}{b^2}-\frac{x^2}{a^2}+\cancel{\left[\frac{xy}{ab}\right]^2}=\cos^2\alpha+\cancel{\left[\frac{xy}{ab}\right]^2}-2\frac{xy}{ab}\cos\alpha$$

Rearrange terms

$$1-\cos^2\alpha=\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}$$

$$\sin^2\alpha=\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\text{RHS.}$$

Hence proved.

- If $\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

$$\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{x-3}{x-4} + \frac{x+3}{x+4}}{1-\left(\frac{x-3}{x-4}\right)\left(\frac{x+3}{x+4}\right)}\right] = \frac{\pi}{4}$$

$$\frac{(x-3)(x+4) + (x+3)(x-4)}{(x-4)(x+4) - (x-3)(x+3)} = \tan\left(\frac{\pi}{4}\right)$$

$$\frac{x^2+x-12 + x^2-x-12}{x^2-16-(x^2-9)} = 1$$

$$\frac{2x^2-24}{-7} = 1, \qquad 2x^2 = 17$$

$$x = \pm\sqrt{\frac{17}{2}}$$

- Prove that $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let, $\sin^{-1}x = \theta$, such that $x = \sin\theta$.

RHS

$$\sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(\sin 3\theta)$$

$$-\frac{1}{2} \le x \le \frac{1}{2}$$

$$-\frac{\pi}{6} \le \sin^{-1}x \le \frac{\pi}{6}$$

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$$

$$-\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

$\therefore$ 3θ is within principal branch.

so, $\sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1}(x) = \text{LHS}$.

Hence proved.

- Find the value: $\sin\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

$$\sin\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right]$$

$$\sin\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right]$$

$$\sin\left[\tan^{-1}\left(\frac{\frac{9+8}{12}}{\frac{6}{12}}\right)\right]$$

$$\sin\left[\tan^{-1}\left(\frac{17}{6}\right)\right]$$

$$\sin\left[\sin^{-1}\left(\frac{17}{\sqrt{325}}\right)\right]$$

$$= \frac{17}{\sqrt{325}}.$$